Bsp: $\det : K^2 \times K^2 \longrightarrow K$

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right) \longmapsto ad - bc$$

$$M(\det) = \boxed{\begin{pmatrix} \det\left(\overset{e_1}{\boxed{\begin{matrix} 1 \\ 0 \end{matrix}}} \overset{e_1}{\boxed{\begin{matrix} 1 \\ 0 \end{matrix}}} \right) & \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \det\left(\underset{e_1}{\boxed{\begin{matrix} 0 \\ 1 \end{matrix}}} \underset{e_2}{\boxed{\begin{matrix} 1 \\ 0 \end{matrix}}} \right) & \det \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\beta_{M(\det)}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right)$$

$$= (a \;\; b) \cdot M(\det) \cdot \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= (a \;\; b) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= (a \;\; b) \cdot \begin{pmatrix} d \\ -c \end{pmatrix} = ad - bc$$

$$= \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \checkmark$$

Beweis:

$\beta_{M(\beta)} = \beta$: $\quad \underline{v} = \sum_i v_i \underline{e}_i, \quad \underline{w} = \sum_j w_j \underline{e}_j \in V$ beliebig

$$\beta_{M(\beta)}(\underline{v}, \underline{w}) = {}^t\underline{v} \cdot M(\beta) \cdot \underline{w}$$
$$= \sum_{i,j} v_i w_j \underbrace{{}^t\underline{e}_i M(\beta) \underline{e}_j}_{M(\beta)_{ij}}$$

$$\left[{}^t\underline{e}_i M \underline{e}_j = \underbrace{(0 \ldots \underset{i}{1}\, 0 \ldots 0)\, M \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_j}_{j\text{-te Spalte von } M}}_{M_{ij}} \right]$$

$$= \sum_{ij} v_i w_j M(\beta)_{ij}$$
$$= \sum_{ij} v_i w_j \beta(\underline{e}_i, \underline{e}_j)$$
$$= \sum_j w_j \beta\Big(\sum_i v_i \underline{e}_i, \underline{e}_j\Big)$$

β bilinear

$$= \beta\Big(\sum_i v_i \underline{e}_i, \sum_j w_j \underline{e}_j\Big)$$
$$= \beta(\underline{v}, \underline{w})$$

$M(\beta_A) = A$:

$$M(\beta_A) = \left(\beta_A(e_i, e_j) \right)_{ij}$$

$$= \big(\underbrace{{}^t e_i \cdot A \cdot e_j}_{A_{ij}} \big)_{ij}$$

$$= A$$

$\square$

<u>Beweis</u>:

$$\left\{\begin{matrix}\text{Bilinearformen} \\ \text{auf } V\end{matrix}\right\} \xrightarrow{\cong} \left\{\begin{matrix}\text{Bilinearformen} \\ \text{auf } K^n\end{matrix}\right\} \xrightarrow{\cong} M(n \times n; K)$$

$$\beta \longmapsto \beta \circ (\Phi_{\mathcal{B}} \times \Phi_{\mathcal{B}})$$

$$\beta \circ (\Phi_{\mathcal{B}}^{-1} \times \Phi_{\mathcal{B}}^{-1}) \longleftarrow\!\shortmid \beta$$

$$\beta \underset{\text{s.o.}}{\longmapsto} M(\beta)$$

$$\beta \longmapsto M(\beta \circ (\Phi_{\mathcal{B}} \times \Phi_{\mathcal{B}})) = M_{\mathcal{B}}(\beta)$$

$\square$

Beweis: ($\Leftarrow$) Sei $A = M_{\mathcal{B}}(\beta)$
$A' = M_{\mathcal{B}'}(\beta)$

darstellende Matrizen von Bilinearf.

$$\begin{tikzcd}
K^n \times K^n \arrow[rrd, "{(\underline{v},\underline{w}) \mapsto {}^t\underline{v} A \underline{w}}"] & & \\
& V \times V \arrow[ul, "\cong"', "\Phi_{\mathcal{B}}^{-1} \times \Phi_{\mathcal{B}}^{-1}"] \arrow[r, "\beta"] & K \\
K^n \times K^n \arrow[uu, dashed, "{(\underline{v},\underline{w}) \mapsto (T\cdot\underline{v},\, T\cdot\underline{w})}"] \arrow[ur, "\cong", "\Phi_{\mathcal{B}'} \times \Phi_{\mathcal{B}'}"'] \arrow[rru, "{(\underline{v},\underline{w}) \mapsto {}^t\underline{v} A' \underline{w}}"'] & &
\end{tikzcd}$$

Def. $T := M(\Phi_{\mathcal{B}}^{-1} \circ \Phi_{\mathcal{B}'})$

darstellende Matrix einer linearen Abb.

Dann gilt für alle $\underline{v}, \underline{w} \in K^n$:

$${}^t\underline{v} A' \underline{w} = {}^t(T\underline{v}) A (T\underline{w}),$$

also $${}^t\underline{v} A' \underline{w} = {}^t\underline{v} \cdot ({}^tT A T) \underline{w}$$

Setze für $\underline{v}, \underline{w}$ $\underline{e}_i, \underline{e}_j$ ein:

$$(A')_{ij} = ({}^tTAT)_{ij}.$$

Also $A' = {}^tTAT$.

($\Rightarrow$) Sei $A' = {}^tTAT$

wähle $V := K^n$

$\beta := \beta_A$

$\mathcal{B} :=$ Standardbasis

$\mathcal{B}' :=$ Spalten von T

$$\begin{tikzcd}
K^n \times K^n \arrow[dr, equal] \arrow[drr, "(v,w) \mapsto {}^tv A w"] & & \\
& K^n \times K^n \arrow[r, "\beta_A"] & K \\
K^n \times K^n \arrow[uu, "T"', "{(Tv,Tw) \,\leftarrow\!\mapsto\, (v,w)}"] \arrow[ur, "\cong", "\Phi_{\mathcal{B}'} \times \Phi_{\mathcal{B}'}"'] \arrow[urr, "(v,w) \mapsto {}^tv A' w"'] & &
\end{tikzcd}$$

$\square$

$$A \in M(n \times n; K)$$

$$K^n \xrightarrow{F_A} K^n$$

$$\underline{v} \longmapsto A \cdot \underline{v}$$

A, A' stellen denselben Endomorphismus dar

$$\Updownarrow$$

$$\exists T: \quad A' = T^{-1} \cdot A \cdot T$$

$$K^n \times K^n \xrightarrow{B_A} K$$

$$(\underline{v}, \underline{w}) \longmapsto {}^t\underline{v}\, A\, \underline{w}$$

A, A' stellen dieselbe Bilinearform dar

$$\Updownarrow$$

$$\exists T: \quad A' = {}^tT \cdot A \cdot T$$

(siehe Notiz zur Determinante
Ende Vorlesung 22)

$K = \mathbb{Z}/2\mathbb{Z} \Rightarrow [1] + [1] = [0]$

$(\Leftarrow)$ $\beta(\underline{v}, \underline{v}) = -\beta(\underline{v}, \underline{v})$, also

$$\beta(\underline{v}, \underline{v}) + \beta(\underline{v}, \underline{v}) = 0$$
$$\underbrace{(1+1)}_{\in K^\times} \cdot \beta(\underline{v}, \underline{v}) = 0 \quad \Big| \; (1+1)^{-1}$$
$$\beta(\underline{v}, \underline{v}) = 0 \quad \Big)$$

<u>Beweisteile</u>: $\mathcal{B} = (\underline{b}_1, \ldots, \underline{b}_n)$,

also $M_{ij} = \beta(\underline{b}_i, \underline{b}_j)$

(symm. $\Rightarrow$)
$$M_{ij} = \beta(\underline{b}_i, \underline{b}_j) = \beta(\underline{b}_j, \underline{b}_i) = M_{ji} = ({}^tM)_{ij}$$

(altern. $\Leftarrow$) Sei ${}^tM = -M$ und $M_{ii} = 0 \ \forall i$.

Sei $\underline{v} \in V$ beliebig

$\underline{v} = \sum_i v_i \underline{b}_i$

$$\beta(\underline{v}, \underline{v}) = \beta\left(\sum_i v_i \underline{b}_i, \sum_i v_i \underline{b}_i\right)$$

$$= \sum_i v_i \, \beta\left(\underline{b}_i, \sum_j v_j \underline{b}_j\right)$$

$$= \sum_{i,j} v_i v_j \underbrace{\beta(\underline{b}_i, \underline{b}_j)}$$

$$= \sum_{ij} v_i v_j M_{ij}$$

$$= \sum_{i,j:\, i>j} v_i v_j M_{ij} + \sum_i v_i^2 \cancel{M_{ii}} + \sum_{i,j:\, i<j} v_i v_j M_{ij}$$

$$\sum_{i,j:\, i<j} v_i v_j (-M_{ji})$$

$$= \sum_{j,i:\, j<i} v_j v_i (-M_{ij})$$

$$= \sum_{i,j:\, i>j} v_i v_j (-M_{ij})$$

$$= 0$$

$\square$